MAT 219 (lecture 12)

- Review of Chapter 7 so far: Repeated Eigenvalues
  e^{At} and Fundamental Matrix Ψ
- §7.9: Nonhomogeneous Linear Systems

Last Time:

Solutions to $\underline{x}' = A\underline{x}$ with repeated eigenvalues $\rightarrow$ generalized eigenvectors

$$\begin{pmatrix}\text{\# of times} \\ \text{that } \lambda \text{ is repeated}\end{pmatrix} = \begin{pmatrix}\text{\# of fundamental} \\ \text{solutions with } e^{\lambda t}\end{pmatrix}$$

Eigenvectors $A\underline{v} = \lambda \underline{v}$ $\rightsquigarrow e^{\lambda t}\underline{v}$

(i.e. $(A-\lambda I)\underline{v} = \underline{0}$)

Gen. Eigenvect $A\underline{w} = \lambda\underline{w} + \underline{v}$ $\rightsquigarrow e^{\lambda t}(\underline{w} + t\underline{v})$

(i.e. $(A-\lambda I)\underline{w} = \underline{v}$)

Gen.^{2} Eigenvect $A\underline{u} = \lambda\underline{u} + \underline{w}$ $\rightsquigarrow e^{\lambda t}\left(\underline{u} + t\underline{w} + \frac{t^2}{2}\underline{v}\right)$

(i.e. $(A-\lambda I)\underline{u} = \underline{w}$)

etc...

General solution = Sum of c_i times (fundamental solutions)

EX: 2-eigenvectors $\underline{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}$

w/ $\begin{bmatrix}3\\2\\1\end{bmatrix}$ - gen. eigenv $\underline{w} = \begin{bmatrix}0\\1\\1\end{bmatrix}$

General solution:

$$\underline{x} = c_1 e^{2t}\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2 e^{2t}\begin{bmatrix}1\\1\\0\end{bmatrix} + c_3 e^{2t}\left(\begin{bmatrix}0\\1\\1\end{bmatrix} + t\begin{bmatrix}3\\2\\1\end{bmatrix}\right)$$

EX: $(3+2i)$-eigenvector $\underline{v} = \begin{bmatrix}1\\0\\1+i\\-i\end{bmatrix}$

$\underline{v}$ - gen. eigenvector $\underline{w} = \begin{bmatrix}0\\1\\1\\-i\end{bmatrix}$ (Too complicated?)

General solution:

$$\left.\begin{aligned}\underline{x} = {} & c_1 e^{3t}\left(\begin{bmatrix}1\\0\\1\\0\end{bmatrix}\cos 2t - \begin{bmatrix}0\\0\\1\\-1\end{bmatrix}\sin 2t\right) \\ & + c_2 e^{3t}\left(\begin{bmatrix}1\\0\\1\\0\end{bmatrix}\sin 2t + \begin{bmatrix}0\\0\\1\\-1\end{bmatrix}\cos 2t\right)\end{aligned}\right\} \underline{v}$$

$$+ c_3 e^{3t}\left(\begin{bmatrix}0\\1\\1\\0\end{bmatrix}\cos 2t - \begin{bmatrix}0\\0\\0\\-1\end{bmatrix}\sin 2t + t\begin{bmatrix}1\\0\\1\\0\end{bmatrix}\cos 2t - t\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}\sin 2t\right)$$

$$+ c_4 e^{3t}\left(\begin{bmatrix}0\\1\\1\\0\end{bmatrix}\sin 2t + \begin{bmatrix}0\\0\\0\\-1\end{bmatrix}\cos 2t + t\begin{bmatrix}1\\0\\1\\0\end{bmatrix}\sin 2t + t\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}\cos 2t\right)$$

More Review:

Has fundamental solutions as columns.

Fundamental Matrix Ψ

and Exponential Matrix e^{At}

Real, Distinct Eigenvalues

($\lambda_1, \lambda_2, \ldots$ with eigenvectors $\underline{v}_1, \underline{v}_2, \ldots$)

$$\Psi = \begin{bmatrix} | & | & \\ \underline{v}_1 & \underline{v}_2 & \cdots \\ | & | & \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & & O \\ & e^{\lambda_2 t} & \\ O & & \ddots \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} | & | & \\ \underline{v}_1 & \underline{v}_2 & \cdots \\ | & | & \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & & O \\ & e^{\lambda_2 t} & \\ O & & \ddots \end{bmatrix}\begin{bmatrix} | & | & \\ \underline{v}_1 & \underline{v}_2 & \cdots \\ | & | & \end{bmatrix}^{-1}$$

Complex Conjugate Eigenvalues

($\lambda = a \pm bi$ with $\underline{v} = \underline{\alpha} \pm \underline{\beta} i$)

$$\Psi = \begin{bmatrix} | & | \\ \underline{\alpha} & \underline{\beta} \\ | & | \end{bmatrix} e^{at}\begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} | & | \\ \underline{\alpha} & \underline{\beta} \\ | & | \end{bmatrix} e^{at}\begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}\begin{bmatrix} | & | \\ \underline{\alpha} & \underline{\beta} \\ | & | \end{bmatrix}^{-1}$$

Repeated Eigenvalues

(λ, λ with eigenvector $\underline{v}$ & generalized $\underline{w}$)

$$\Psi = \begin{bmatrix} | & | \\ \underline{v} & \underline{w} \\ | & | \end{bmatrix}\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} | & | \\ \underline{v} & \underline{w} \\ | & | \end{bmatrix}\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}\begin{bmatrix} | & | \\ \underline{v} & \underline{w} \\ | & | \end{bmatrix}^{-1}$$

These combine together in "blocks"

EX: If A is 4×4 with

eigenvalues $\lambda = 1 \pm 2i, 3, 3$

$(1+2i)$-eigenv. $\quad \underline{\alpha} + \underline{\beta} i$

3-eigenv. $\quad \underline{v}$

$\underline{v}$-gen. eigenv $\quad \underline{w}$

Then

$$\Psi = \begin{bmatrix} | & | & | & | \\ \underline{\alpha} & \underline{\beta} & \underline{v} & \underline{w} \\ | & | & | & | \end{bmatrix}\begin{bmatrix} e^{t}\cos 2t & e^{t}\sin 2t & 0 & 0 \\ -e^{t}\sin 2t & e^{t}\cos 2t & 0 & 0 \\ 0 & 0 & e^{3t} & te^{3t} \\ 0 & 0 & 0 & e^{3t} \end{bmatrix}$$

"blocks" go on diagonal of eigenval. matrix

This is a good way to think of repeated $\mathbb{C}$ case also.

EX: A 4×4 w/ $\lambda = \pm 2i, \pm 2i$

eigenvector $\quad \underline{\alpha} \pm \underline{\beta} i$

gen. eigenv. $\quad \underline{\gamma} \pm \underline{\delta} i$

(t times diagonal "block" for $e^{\pm 2it}$)

Then

$$\Psi = \begin{bmatrix} | & | & | & | \\ \underline{\alpha} & \underline{\beta} & \underline{\gamma} & \underline{\delta} \\ | & | & | & | \end{bmatrix}\begin{bmatrix} \cos 2t & \sin 2t & t\cos 2t & t\sin 2t \\ -\sin 2t & \cos 2t & -t\sin 2t & t\cos 2t \\ 0 & 0 & \cos 2t & \sin 2t \\ 0 & 0 & -\sin 2t & \cos 2t \end{bmatrix}$$

EX: Use e^{At} to write solution to

$$\underline{x}' = \begin{bmatrix}-3 & 2\\ 1 & -4\end{bmatrix}\underline{x} \quad \text{with} \quad \underline{x}(0) = \begin{bmatrix}1\\4\end{bmatrix}$$

Recall: Solution is $\underline{x} = e^{At}\cdot \underline{x}(0)$

Eigenvalues: $\lambda^2 - (-7)\lambda + (12-2) = 0$

$$(\lambda+2)(\lambda+5) = 0 \qquad \boxed{\lambda = -2, -5}$$

(−2)-Eigenvect: $\begin{bmatrix}-3-(-2) & 2\\ 1 & -4-(-2)\end{bmatrix}\underline{v} = \underline{0} \rightsquigarrow \underline{v} = \boxed{\begin{bmatrix}2\\+1\end{bmatrix}}$

(−5)-Eigenvect: $\begin{bmatrix}-3-(-5) & 2\\ 1 & -4-(-5)\end{bmatrix}\underline{v} = \underline{0} \rightsquigarrow \underline{v} = \begin{bmatrix}2\\-2\end{bmatrix} \xrightarrow{\times\frac{1}{2}} \boxed{\begin{bmatrix}1\\-1\end{bmatrix}}$

Solution:

$$\underline{x} = \underbrace{\begin{bmatrix}2 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}e^{-2t} & 0\\ 0 & e^{-5t}\end{bmatrix}}_{\Psi}\underbrace{\begin{bmatrix}2 & 1\\ 1 & -1\end{bmatrix}^{-1}\begin{bmatrix}1\\4\end{bmatrix}}_{\begin{bmatrix}c_1\\c_2\end{bmatrix}}$$

EX: Use e^{At} to write solution to

$$\underline{x}' = \begin{bmatrix}-6 & 5\\ -5 & 2\end{bmatrix}\underline{x} \quad \text{with} \quad \underline{x}(0) = \begin{bmatrix}1\\2\end{bmatrix}$$

Eigenvalues: $\lambda^2 - (-4)\lambda + (-12+25) = 0$

$$(\lambda+2)^2 + 9 = 0 \qquad \boxed{\lambda = -2\pm 3i}$$

$(-2\pm 3i)$ Eigenv: $\begin{bmatrix}-6-(-2+3i) & 5\\ -5 & 2-(-2+3i)\end{bmatrix}\underline{v} = \underline{0}$

$$\underline{v} = \boxed{\begin{bmatrix}5\\ +4+3i\end{bmatrix}} = \begin{bmatrix}5\\4\end{bmatrix} + \begin{bmatrix}0\\3\end{bmatrix}i$$

Solution:

$$\underline{x} = \underbrace{\begin{bmatrix}5 & 0\\ 4 & 3\end{bmatrix}e^{-2t}\begin{bmatrix}\cos 3t & \sin 3t\\ -\sin 3t & \cos 3t\end{bmatrix}}_{\Psi}\underbrace{\begin{bmatrix}5 & 0\\ 4 & 3\end{bmatrix}^{-1}\begin{bmatrix}1\\2\end{bmatrix}}_{\begin{bmatrix}c_1\\c_2\end{bmatrix}}$$

Homework:

- $\underline{x}' = \begin{bmatrix}-1 & 2\\ 3 & -2\end{bmatrix}\underline{x}$ with $\underline{x}(0) = \begin{bmatrix}-1\\0\end{bmatrix}$
- $\underline{x}' = \begin{bmatrix}4 & -2\\ -6 & 3\end{bmatrix}\underline{x}$ with $\underline{x}(0) = \begin{bmatrix}1\\1\end{bmatrix}$

Homework:

- $\underline{x}' = \begin{bmatrix}5 & 4\\ -5 & -3\end{bmatrix}\underline{x}$ with $\underline{x}(0) = \begin{bmatrix}2\\1\end{bmatrix}$
- $\underline{x}' = \begin{bmatrix}3 & 9\\ -2 & -3\end{bmatrix}\underline{x}$ with $\underline{x}(0) = \begin{bmatrix}1\\2\end{bmatrix}$

EX: Use e^{At} to write solution to

$$\underline{x}' = \begin{bmatrix} 8 & 4 \\ -1 & 4 \end{bmatrix} \underline{x} \quad \text{with} \quad \underline{x}(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Eigenvalues: $\lambda^2 - 12\lambda + 36 = 0$

$(\lambda - 6)^2 = 0 \qquad \boxed{\lambda = 6, 6}$

6-Eigenvect: $\begin{bmatrix} 8-6 & 4 \\ -1 & 4-6 \end{bmatrix} \underline{v} = \underline{0} \rightsquigarrow \underline{v} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} \xrightarrow{\times \frac{1}{2}} \boxed{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}$

$\begin{bmatrix} 2 \\ -1 \end{bmatrix}$-Gen. 6-Eigenv: $\begin{bmatrix} 8-6 & 4 \\ -1 & 4-6 \end{bmatrix} \underline{w} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \underline{w} = \boxed{\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}}$

(Note: $\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is also a gen. eigenv.)

Solution:

$$\underline{x} = \underbrace{\begin{bmatrix} 2 & 0 \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} e^{6t} & te^{6t} \\ 0 & e^{6t} \end{bmatrix}}_{\Psi} \underbrace{\begin{bmatrix} 2 & 0 \\ -1 & \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 4 \end{bmatrix}}_{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}$$

Homework:

- $\underline{x}' = \begin{bmatrix} 3 & 3 \\ -3 & 3 \end{bmatrix} \underline{x}$ with $\underline{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$
- $\underline{x}' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \underline{x}$ with $\underline{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Final Topic: §7.9. Nonhomogeneous Equations (Variation of Parameters)

Recall: Nonhomogeneous system is

$$\underline{x}' = A\underline{x} + \underline{g}(t)$$

(Nonzero vector function of t)

For example,

$$\begin{cases} x' = 2x - y + 3t \\ y' = x + y - e^{2t} \end{cases}$$

$$\underline{x}' = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \underline{x} + \underbrace{\begin{bmatrix} 3t \\ -e^{2t} \end{bmatrix}}_{\underline{g}(t)}$$

There are (at least) four different methods to solve $\underline{x}' = A\underline{x} + \underline{g}$

1. Substitute $\underline{x} = P\underline{y}$ to change system to be triangular
2. "Undetermined Coefficients"
3. Laplace Transforms
4. $\boxed{\text{"Variation of Parameters"}}$

We will learn only Variation of Parameters

→ It is the simplest to explain, but the most terrible to compute.

Idea: The general solution to homog. system $\underline{x}' = A\underline{x}$ is

$$\underline{x} = \Psi \cdot \underline{c} \leftarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \text{ the unknown constants}$$

→ Write solution to nonhomogeneous system $\underline{x}' = A\underline{x} + \underline{g}$ as

$$\underline{x} = \Psi \cdot \underline{u}(t) \leftarrow \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \text{ replace constants by functions!}$$

(i.e. Allow the constants — the "parameters" — to "vary"... Varying Parameters)

Plug $\underline{x} = \Psi \cdot \underline{u}$ into system $\underline{x}' = A\underline{x} + \underline{g}$.

product rule

$$(\Psi \cdot \underline{u})' = A(\Psi \cdot \underline{u}) + \underline{g}$$

$$\boxed{\Psi' \cdot \underline{u}} + \Psi \cdot \underline{u}' = \boxed{(A\Psi) \cdot \underline{u}} + \underline{g} \qquad \Psi' = A\Psi$$

cancel

$$\underline{u}' = \Psi^{-1} \cdot \underline{g}$$

Variation of Parameters Formula:

$\underline{x}' = A\underline{x} + \underline{g}$ has solution

$$\boxed{\underline{x} = \underbrace{\Psi \cdot \underline{c}}_{\text{solution to } \underline{x}' = A\underline{x}} + \Psi \cdot \Big(\underbrace{\int \Psi^{-1} \cdot \underline{g}\, dt}_{\text{anti-derivative}}\Big)}$$

where Ψ is the fundamental matrix for $\underline{x}' = A\underline{x}$

EX: Solve $\underline{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\underline{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$

First we compute Ψ: solution to $\underline{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\underline{x}$

Eigenvals

$$\lambda^2 + 4\lambda + 3 = 0$$
$$(\lambda+1)(\lambda+3) = 0 \qquad \boxed{\lambda = -1, -3}$$

(-1) Eigenvect $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\underline{v} = \underline{0} \rightsquigarrow \underline{v} = \boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

(-3) Eigenvect $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\underline{v} = \underline{0} \rightsquigarrow \underline{v} = \boxed{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

$$\Psi = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} = \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix}$$

(EX continues)

Now compute $\Psi\left(\int \Psi^{-1} g \, dt\right)$:

$$\Psi^{-1} = \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix}^{-1} = \frac{1}{e^{-t}(-e^{-3t}) - (e^{-3t})e^{-t}} \begin{bmatrix} -e^{-3t} & -e^{-3t} \\ -e^{-t} & e^{-t} \end{bmatrix}$$

(Recall: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$)

$$= -\frac{1}{2} e^{4t} \begin{bmatrix} -e^{-3t} & -e^{-3t} \\ -e^{-t} & e^{-t} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{t} & e^{t} \\ e^{3t} & -e^{3t} \end{bmatrix}$$

$$\Psi^{-1} g = \frac{1}{2}\begin{bmatrix} e^{t} & e^{t} \\ e^{3t} & -e^{3t} \end{bmatrix}\begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 + 3te^{t} \\ 2e^{2t} - 3te^{3t} \end{bmatrix}$$

$$\int \Psi^{-1} g \, dt = \int \frac{1}{2}\begin{bmatrix} 2 + 3te^{t} \\ 2e^{2t} - 3te^{3t} \end{bmatrix} dt \xrightarrow{\text{int. by parts}} \frac{1}{2}\begin{bmatrix} 2t + (3te^{t} - 3e^{t}) \\ e^{2t} - (te^{3t} - \frac{1}{3}e^{3t}) \end{bmatrix}$$

$$\Psi\left(\int \Psi^{-1} g \, dt\right) = \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 2t + 3te^{t} - 3e^{t} \\ e^{2t} - te^{3t} + \frac{1}{3}e^{3t} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} (2te^{-t} + 3t - 3) + (e^{-t} - t + \frac{1}{3}) \\ (2te^{-t} + 3t - 3) - (e^{-t} - t + \frac{1}{3}) \end{bmatrix}$$

Answer:

$$\underline{x} = c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} (2t+1)e^{-t} + 2t - \frac{8}{3} \\ (2t-1)e^{-t} + 3t - \frac{10}{3} \end{bmatrix}$$

EX: Solve $\underline{x}' = \begin{bmatrix} 2 & 1 \\ -4 & 6 \end{bmatrix}\underline{x} + \begin{bmatrix} e^{4t} \\ 1 \end{bmatrix}$ with $\underline{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(This won't be pleasant...)

Compute Ψ: $\underline{x}' = \begin{bmatrix} 2 & 1 \\ -4 & 6 \end{bmatrix}\underline{x}$

Eigenvalues. $\lambda^2 - 8\lambda + 16 = 0$

$(\lambda - 4)^2 = 0$ $\boxed{\lambda = 4, 4}$

4 Eigenvect. $\begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}\underline{v} = \underline{0} \rightsquigarrow \underline{v} = \boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ Gen 4 Eigenv $\begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}\underline{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightsquigarrow \underline{w} = \boxed{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$

$$\Psi = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} e^{4t} & te^{4t} \\ 0 & e^{4t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{4t} & te^{4t} \\ 2e^{4t} & (2t+1)e^{4t} \end{bmatrix}$$

Note: Horrible terms will always cancel.

Compute $\Psi \int \Psi^{-1} g$:

$$\Psi^{-1} = \frac{1}{(\cancel{2t}+1)e^{8t} - \cancel{2te^{8t}}}\begin{bmatrix} (2t+1)e^{4t} & -te^{4t} \\ -2e^{4t} & e^{4t} \end{bmatrix}$$

$$= \begin{bmatrix} (2t+1)e^{-4t} & -te^{-4t} \\ -2e^{-4t} & e^{-4t} \end{bmatrix}$$

(EX continues)

$$\Psi^{-1} g = \begin{bmatrix} (2t+1)e^{-4t} & -te^{-4t} \\ -2e^{-4t} & e^{-4t} \end{bmatrix} \begin{bmatrix} e^{4t} \\ 1 \end{bmatrix}$$

Note: I chose g for this example very carefully

$$= \begin{bmatrix} (2t+1) - te^{-4t} \\ -2 + e^{-4t} \end{bmatrix}$$

$$\int \Psi^{-1} g \, dt = \int \begin{bmatrix} 2t+1 - te^{-4t} \\ -2 + e^{-4t} \end{bmatrix} dt$$

int. by parts

$$= \begin{bmatrix} t^2 + t - \left(-\tfrac{1}{4} te^{-4t} - \tfrac{1}{16} e^{-4t}\right) \\ -2t - \tfrac{1}{4} e^{-4t} \end{bmatrix}$$

$$\Psi \int \Psi^{-1} g \, dt = \begin{bmatrix} e^{4t} & te^{4t} \\ 2e^{4t} & (2t+1)e^{4t} \end{bmatrix} \begin{bmatrix} t^2 + t + \tfrac{1}{4} te^{-4t} + \tfrac{1}{16} e^{-4t} \\ -2t - \tfrac{1}{4} e^{-4t} \end{bmatrix}$$

$$= \begin{bmatrix} \left((t^2+t)e^{4t} + \tfrac{1}{4}t + \tfrac{1}{16}\right) + \left(-2t^2 e^{4t} - \tfrac{1}{4}\right) \\ \left(2(t^2+t)e^{4t} + \tfrac{1}{2}t + \tfrac{1}{8}\right) + \left(-(4t^2+2t)e^{4t} - \tfrac{2t+1}{4}\right) \end{bmatrix}$$

$$= \begin{bmatrix} (-t^2+t)e^{4t} + \tfrac{1}{4}t - \tfrac{3}{16} \\ -2t^2 e^{4t} - \tfrac{1}{8} \end{bmatrix}$$

(EX continues)

General Solution:

$$\underline{x} = c_1 e^{4t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} t \\ 2t+1 \end{bmatrix} + \begin{bmatrix} (-t^2+t)e^{4t} + \tfrac{1}{4}t - \tfrac{3}{16} \\ -2t^2 e^{4t} - \tfrac{1}{8} \end{bmatrix}$$

→ Now we can plug in initial values

$$\underline{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \underline{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -3/16 \\ -1/8 \end{bmatrix}$$

$$\begin{bmatrix} 3/16 \\ 9/8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \rightsquigarrow \underline{c_1 = 3/16}$$

$$9/8 = 2 \cdot 3/16 + c_2$$

$$c_2 = 6/8 = \underline{\underline{3/4}}$$

Answer:

$$\underline{x} = \tfrac{3}{16} e^{4t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \tfrac{3}{4} e^{4t} \begin{bmatrix} t \\ 2t+1 \end{bmatrix} + \begin{bmatrix} (-t^2+t)e^{4t} + \tfrac{1}{4}t - \tfrac{3}{16} \\ 2t^2 e^{4t} - \tfrac{1}{8} \end{bmatrix}$$

$$= \begin{bmatrix} \left(-t^2 + \tfrac{7}{4}t + \tfrac{3}{16}\right)e^{4t} + \tfrac{1}{4}t - \tfrac{3}{16} \\ \left(2t^2 + \tfrac{3}{2}t + \tfrac{9}{8}\right)e^{4t} - \tfrac{1}{8} \end{bmatrix}$$

FINISHED!!!

(you can stop crying now...)

Please do not check this computation for mistakes...